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## TOROIDAL SURGERIES ON HYPERBOLIC KNOTS

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**ABSTRACT.** For a hyperbolic knot  $K$  in  $S^3$ , a toroidal surgery is Dehn surgery which yields a 3-manifold containing an incompressible torus. It is known that a toroidal surgery on  $K$  is an integer or a half-integer. In this paper, we prove that all integers occur among the toroidal slopes of hyperbolic knots with bridge index at most three and tunnel number one.

### 1. INTRODUCTION

Let  $K$  be a knot in the 3-sphere  $S^3$ , and let  $E(K) = S^3 - \text{Int}N(K)$  be its exterior. A *slope* on  $\partial E(K)$  is the isotopy class of an essential unoriented simple loop. As usual [11], the set of slopes on  $\partial E(K)$  is parameterized by  $\mathbb{Q} \cup \{\infty\}$  so that  $1/0$  is the meridian slope and  $0/1$  is the longitude slope. For a slope  $r$  on  $\partial E(K)$ ,  $K(r)$  denotes the closed orientable 3-manifold obtained by *r-Dehn surgery* on  $K$ . Thus  $K(r) = E(K) \cup V$ , where  $V$  is a solid torus glued to  $E(K)$  along their boundaries in such a way that  $r$  bounds a disk in  $V$ .

Now suppose that  $K$  is a hyperbolic knot, i.e. the interior of  $E(K)$  has a complete hyperbolic structure. If  $K(r)$  is not hyperbolic, the surgery and the slope  $r$  are said to be *exceptional*. By the hyperbolic Dehn surgery theorem [12],  $K$  has only finitely many exceptional surgeries. A closed 3-manifold is *toroidal* if it contains an incompressible torus. If  $K(r)$  is toroidal, the surgery is said to be *toroidal*. Clearly, a toroidal surgery is exceptional.

There are some results on toroidal surgeries on hyperbolic knots. Gordon and Luecke [8] showed that if  $K(m/n)$  is toroidal, then  $|n| \leq 2$ . Hence a toroidal slope on a hyperbolic knot is either an integer or a half-integer. For hyperbolic alternating knots, toroidal slopes are integers divisible by four [1] (see also [10]). In this paper, we show that all integers can occur among the toroidal slopes of hyperbolic knots.

**Theorem 1.1.** *For any integer  $r$ , there exists a hyperbolic knot  $K$  in  $S^3$  such that  $K(r)$  is toroidal. Furthermore,  $K$  has bridge index at most three and tunnel number one.*

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## 2. DOUBLY SEIFERT-FIBERED KNOTS

In this section, we will describe a construction of knots in  $S^3$  that have toroidal surgeries done by Dean [4].

Let  $H$  be a standardly embedded handlebody of genus two in  $S^3$ . Then  $H' = S^3 - \text{Int}H$  is also a handlebody of genus two. Let  $F = \partial H = \partial H'$ . If a knot  $K$  is embedded in  $F$ , then  $\partial N(K) \cap F$  defines a slope on  $\partial E(K)$ , which is called the *surface slope of  $K$  with respect to  $F$* . Note that a surface slope is always integral.

**Lemma 2.1.** *Let  $K$  be a knot in  $F$ , and let  $r$  be the surface slope of  $K$  with respect to  $F$ . Assume that  $K$  is non-separating in  $F$ . Then  $K(r) \cong W \cup W'$ , where  $W$  ( $W'$ ) is obtained from  $H$  ( $H'$  resp.) by attaching a 2-handle along  $K$ , and they are glued along their boundaries  $\partial W$  and  $\partial W'$ , which are tori.*

*Proof.* This is a special case of [4, Lemma 2.1.1]. Let  $c_1$  and  $c_2$  be the curves  $F \cap \partial N(K)$ . Then  $c_i$  bounds a meridian disk  $D_i$  of the attached solid torus  $V$  in  $K(r)$  for  $i = 1, 2$ . Let  $\hat{F} = (F - N(K)) \cup D_1 \cup D_2$ . Since  $K$  is non-separating in  $F$ ,  $\hat{F}$  is a torus. We split  $K(r)$  along  $\hat{F}$  into  $W$  and  $W'$ . Then  $W$  and  $W'$  are homeomorphic to the described ones.  $\square$

For non-zero integers  $m$  and  $n$ , let  $G_{m,n}$  denote the group  $\langle x, y \mid x^m y^n = 1 \rangle$ . An element in a free group is *primitive* if it is a part of a basis. An element  $w$  in the free group  $\langle x, y \rangle$  is  $(m, n)$  *Seifert-fibered* if  $\langle x, y \mid w = 1 \rangle \cong G_{m,n}$ . If  $|m| = 1$  or  $|n| = 1$ , then  $G_{m,n} \cong \mathbb{Z}$ .

If a knot  $K$  in  $F$  represents a Seifert-fibered element of  $\pi_1(H)$ , then we say that  $K$  is *Seifert-fibered* with respect to  $H$ . In particular, if  $K$  represents a primitive element of  $\pi_1(H)$ , then  $K$  is said to be *primitive* with respect to  $H$ . Also, if  $K$  is Seifert-fibered with respect to both of  $H$  and  $H'$ , then it is said to be *doubly Seifert-fibered* with respect to  $F$ . Note that the abelianization of  $G_{m,n}$  is  $\mathbb{Z} \oplus \mathbb{Z}_{(m,n)}$ , where  $(m, n)$  is the greatest common divisor of  $m$  and  $n$ . Therefore if  $K$  is Seifert-fibered with respect to  $H$ , say, then  $K$  is non-separating on  $F = \partial H$ .

**Lemma 2.2.** *If a knot  $K$  on  $\partial H$  is  $(m, n)$  Seifert-fibered with respect to  $H$  for  $|m|, |n| \geq 2$ , then the manifold  $W$  obtained by adding a 2-handle to  $H$  along  $K$  is a Seifert fibered manifold with incompressible boundary.*

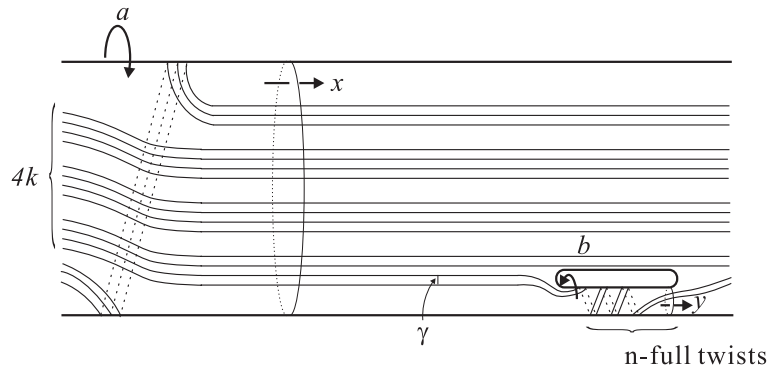
*Proof.* Note that  $W$  has Heegaard genus two. By additivity of Heegaard genus (see [3]),  $W$  is irreducible, since  $\pi_1(W) = G_{m,n}$ . Hence  $W$  is Haken. Then  $W$  is a Seifert fibered manifold by [13], since  $G_{m,n}$  has a nontrivial center. The last part follows from the fact that the only Seifert fibered manifold with non-empty compressible boundary is a solid torus. See also [4, Lemma 2.2.1].  $\square$

**Lemma 2.3.** *Let  $K$  be a doubly Seifert-fibered knot in  $F$  with surface slope  $r$ . Then  $K(r)$  is toroidal.*

*Proof.* This immediately follows from Lemmas 2.1 and 2.2.  $\square$

## 3. PROOF OF THEOREM 1.1

Let  $r$  be an integer. If  $r$  is a toroidal surgery on a knot  $K$ , then  $-r$  is one on the mirror image of  $K$ . Therefore we may assume that  $r \geq 0$ .

FIGURE 1.  $K(4k+3, 3, 2, n)$ 

We will divide the proof into three cases.

*Case 1.*  $r \equiv 0 \pmod{4}$ .

Let  $K(b_1, b_2)$  be the 2-bridge knot corresponding to a continued fraction  $[b_1, b_2]$ . Then  $K(2, -2)$  is the figure-eight knot, and 0 and 4 are toroidal surgeries [12]. If  $r \geq 8$ ,  $K(3, r/2)$  is hyperbolic, and  $r$  gives a toroidal surgery [2, Theorem 1.1(2)]. Note that any 2-bridge knot has tunnel number one.

*Case 2.*  $r \equiv \pm 1 \pmod{4}$ .

Let  $k \geq 1, k \not\equiv 0 \pmod{3}$  and  $n \leq -2$ . Let  $K(4k+3, 3, 2, n)$  be the twisted torus knot lying on  $H$ . It is obtained from the torus knot of type  $(4k+3, 3)$  by adding  $n$ -full twists on two strands that are parallel in the standard torus knot picture [4]. See Figure 1, where the two ends are glued to form  $H$ . We see that the surface slope with respect to  $\partial H$  is  $3(4k+3) + 4n$ . As a knot in  $S^3$ , it is isotopic to  $K(3, 4k+3, 2, n)$ , and hence it has bridge index at most three. Also, it is clear that the arc  $\gamma$  shown in Figure 1 is an unknotting tunnel.

Let  $\{x, y\}$  and  $\{a, b\}$  be the bases of  $\pi_1(H)$  and  $\pi_1(H')$ , respectively, as in Figure 1. The following two lemmas are checked straightforwardly.

**Lemma 3.1.** *In  $\pi_1(H)$ ,*

$$K(4k+3, 3, 2, n) \text{ represents } \begin{cases} x^{\frac{8k+7}{3}} y x^{\frac{4k+2}{3}} y & \text{if } k \equiv 1 \pmod{3}, \\ x^{\frac{8k+5}{3}} y x^{\frac{4k+4}{3}} y & \text{if } k \equiv 2 \pmod{3}. \end{cases}$$

**Lemma 3.2.** *In  $\pi_1(H')$ ,  $K(4k+3, 3, 2, n)$  represents  $a^2 b^{-n} a b^{-n}$ .*

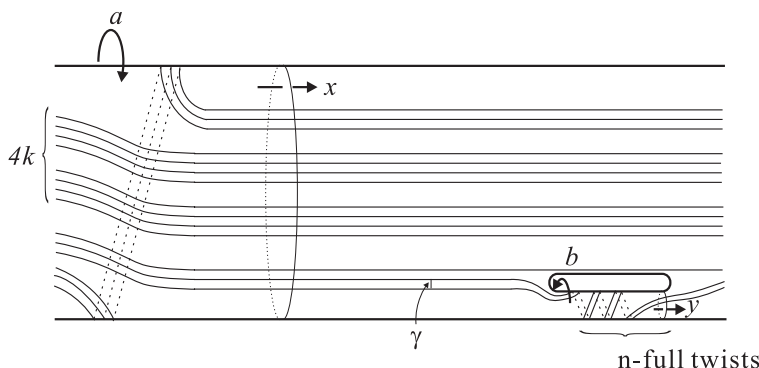
**Lemma 3.3.** *With respect to  $H$ ,*

$$K(4k+3, 3, 2, n) \text{ is } \begin{cases} (\frac{4k+5}{3}, 2) \text{ Seifert-fibered} & \text{if } k \equiv 1 \pmod{3}, \\ (\frac{4k+1}{3}, 2) \text{ Seifert-fibered} & \text{if } k \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* We prove the case where  $k \equiv 1 \pmod{3}$ . The other case is similar.

$$\begin{aligned} \langle x, y \mid x^{\frac{8k+7}{3}} y x^{\frac{4k+2}{3}} y = 1 \rangle &= \langle x, y \mid x^{\frac{4k+5}{3}} (x^{\frac{4k+2}{3}} y)^2 = 1 \rangle \\ &= \langle x, y, z \mid x^{\frac{4k+5}{3}} z^2 = 1, z = x^{\frac{4k+2}{3}} y \rangle \\ &= \langle x, z \mid x^{\frac{4k+5}{3}} z^2 = 1 \rangle. \end{aligned}$$

□

FIGURE 2.  $K(4k+6, 3, 2, n)$ 

**Lemma 3.4.**  $K(4k+3, 3, 2, n)$  is  $(3, n)$  Seifert-fibered with respect to  $H'$ .

*Proof.*

$$\begin{aligned}
 \langle a, b \mid a^2 b^{-n} a b^{-n} = 1 \rangle &= \langle a, b \mid a(ab^{-n})^2 = 1 \rangle \\
 &= \langle a, b, c \mid ac^2 = 1, c = ab^{-n} \rangle \\
 &= \langle b, c \mid cb^n c^2 = 1 \rangle \\
 &= \langle b, c \mid c^3 b^n = 1 \rangle.
 \end{aligned}$$

□

**Proposition 3.5.** For  $K(4k+3, 3, 2, n)$ , the surface slope  $3(4k+3)+4n$  with respect to  $F$  is toroidal. Therefore the knot is hyperbolic.

*Proof.* The knot  $K(4k+3, 3, 2, n)$  is doubly Seifert-fibered by Lemmas 3.3 and 3.4. Hence the surface slope is toroidal by Lemma 2.3. Since it has bridge index at most three, it is either a torus knot or a hyperbolic knot. But a torus knot has no non-zero toroidal surgery. (In fact, 0-surgery on the trefoil is the only toroidal surgery for torus knots. See [9].) Thus the knot is hyperbolic. □

For a given integer  $r > 0$  such that  $r \equiv 1 \pmod{4}$ , we choose  $k$  so that  $r \leq 12k+1, k \not\equiv 0 \pmod{3}$ . Then the knot  $K(4k+3, 3, 2, n)$ ,  $n = -2 - \frac{(12k+1)-r}{4}$ , has the surface slope  $r$  exactly. If  $r \equiv -1 \pmod{4}$ , then consider the knot  $K(7, 3, 2, n)$  with  $n = -6 + \frac{3-r}{4}$ . Then it has the surface slope  $-r$ . Hence its mirror image is a desired one.

As stated before, if a hyperbolic 2-bridge knot has a toroidal slope, then the slope is an integer divisible by 4 [1]. Therefore our knots are 3-bridge.

*Case 3.*  $r \equiv 2 \pmod{4}$ .

Let  $k \geq 1, k \not\equiv 0 \pmod{3}$  and  $n \leq -2$ . Let  $K(4k+6, 3, 2, n)$  be the twisted torus knot lying on  $H$ . See Figure 2. We see that the surface slope with respect to  $\partial H$  is  $3(4k+6) + 4n$ .

As in Case 2, the knot has bridge index at most three, and tunnel number one. We use the same bases of  $\pi_1(H)$  and  $\pi_1(H')$  as in Case 1.

The following two lemmas are checked straightforwardly.

**Lemma 3.6.** In  $\pi_1(H)$ ,

$$K(4k+6, 3, 2, n) \text{ represents } \begin{cases} x^{\frac{8k+13}{3}} y x^{\frac{4k+5}{3}} y & \text{if } k \equiv 1 \pmod{3}, \\ x^{\frac{8k+11}{3}} y x^{\frac{4k+7}{3}} y & \text{if } k \equiv 2 \pmod{3}. \end{cases}$$

**Lemma 3.7.** In  $\pi_1(H')$ ,  $K(4k+6, 3, 2, n)$  represents  $a^2 b^{-n} a b^{-n}$ .

The next lemmas are proved by the same way as in the proofs of Lemmas 3.3 and 3.4.

**Lemma 3.8.** With respect to  $H$ ,

$$K(4k+6, 3, 2, n) \text{ is } \begin{cases} (\frac{4k+8}{3}, 2) \text{ Seifert-fibered} & \text{if } k \equiv 1 \pmod{3}, \\ (\frac{4k+4}{3}, 2) \text{ Seifert-fibered} & \text{if } k \equiv 2 \pmod{3}. \end{cases}$$

**Lemma 3.9.**  $K(4k+6, 3, 2, n)$  is  $(3, n)$  Seifert-fibered with respect to  $H'$ .

**Proposition 3.10.** For  $K(4k+6, 3, 2, n)$ , the surface slope  $3(4k+6) + 4n$  with respect to  $F$  is toroidal. Therefore the knot is hyperbolic.

*Proof.* The arguments in the proof of Proposition 3.5 work well.  $\square$

For a given integer  $r > 0$  such that  $r \equiv 2 \pmod{4}$ , we choose  $k$  so that  $r \leq 12k+10, k \not\equiv 0 \pmod{3}$ . Then the knot  $K(4k+6, 3, 2, n)$ ,  $n = -2 - \frac{(12k+10)-r}{4}$ , has the surface slope  $r$ . As in Case 2, the knot is 3-bridge.

Thus we have proved Theorem 1.1.

*Remark 3.11.* Eudave-Muñoz [5] gave infinitely many hyperbolic knots with half-integer toroidal surgeries. For example, the  $(-2, 3, 7)$ -pretzel knot has a toroidal slope  $37/2$ . Among his knots,  $k(l, m, n, 0)$  (in his notation) has a non-integral toroidal slope

$$-\frac{1}{2} - l + l^2 m + 2lm - 2l^2 m^2 + (2lm - 1)^2 n.$$

Indeed,  $k(3, 1, 1, 0)$  is the  $(-2, 3, 7)$ -pretzel knot. Also,  $k(l, m, 0, p)$  has a non-integral toroidal slope

$$-\frac{1}{2} - l + l^2 m + 2lm - 2l^2 m^2 + (2lm - 1 - l)^2 p.$$

See also [6, Propositions 5.3, 5.4]. Since his knots are expected to give all hyperbolic knots with non-integral toroidal surgeries [7], it seems to be reasonable to conjecture that not all  $n/2$  can be realized as toroidal slopes of hyperbolic knots. In fact, we may conjecture that  $|n/2| \geq 37/2$ .

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